VARIOUS RESULTS ON THE WELL-COVERED DIMENSION OF A GRAPH

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Abstract. The well-covered dimension of various individual graphs and graph families are computed. Special families, such as the crown graphs, are studied and they yield well-covered dimensions that change depending on the characteristic of the field chosen as the field of scalars. Also, formulas to find the well-covered dimension of graphs obtained by vertex blowups on a known graph, and to the lexicographic product of two known graphs are given.

1. introduction

In this paper, a graph is understood to be undirected and have no loops or multiple edges. While graphs with multiple edges could be taken under consideration, it is not necessary to do so as multiple edges do not add any difficulty or important properties.

A set of vertices in a graph G is said to be independent if no two vertices in the set are joined by an edge. An independent set M of G is called maximal if no independent set of G properly contains M. The largest (in terms of cardinality) maximal independent set (or sets) of G is called a maximum independent set of G, and a graph is said to be well-covered if every maximal independent set of G is also maximum. A well-covered graph could also be defined by the property of all maximal independent sets having the same cardinality. This notion was introduced by Plummer in [4]. In [1], Brown and Nowakowski defined a well-covered weighting of a graph G as a function $w : V(G) \to \mathbf{F}$ that assigns values to the vertices of G in such a way that $\sum_{x\in M} w(x)$ is a constant for all maximal independent sets M of G. It is immediate from the latter definition that one could re-define well-coveredness by saying that a well-covered graph is a graph that admits the constant function equal to 1 as a well-covered weighting of G. We will use Brown and Nowakowski's presentation (notation, nomenclature, etc), although this problem was originally introduced by Caro, Ellingham, Ramey, and Yuster in [2] and [3].

It is easy to show that, once a field $\mathbf{F}$ is fixed, the set of all well-covered weightings of a graph G is an $\mathbf{F}$-vector space, which is called *the well-covered space of* G. The dimension of this vector space over $\mathbf{F}$ is called the *well-covered dimension of* G and is denoted by $wcdim(G, \mathbf{F})$. If $wcdim(G, \mathbf{F})$ does not depend on the field used then the well-covered dimension of G is instead denoted as $wcdim(G)$. The well-covered dimension of a graph G can therefore be considered to be the cardinality of a set of vertices S of G whose weights are independent of each other and on which the weights of the remaining vertices of G are dependent, provided that the weighting is a well-covered weighting. Most of the results obtained in this paper

The first author thanks the Department of Mathematics at CSU, Fresno for its support during his Summer REU.

are independent of the characteristic of $\mathbf{F}$, and when the characteristic becomes something to consider we will be careful to remark on it.

Our graph theoretic notation, algebraic notation, and matrix theoretic notation is standard; the reader can look at [5] for any concepts we fail to define. The vertex set of a graph G is denoted by $V(G)$. The cardinality of a set of vertices V is denoted by $|V|$. A field with $q = p^h$ (p prime) elements is denoted by $\mathbf{F}_q$. The $n \times n$ identity matrix is denoted by I_n. The $n \times n$ matrix where each entry is a 1 is denoted by J_n. An $m \times 1$ column vector where each entry is a 1 is denoted by $\mathbf{1}_m$. An $m \times 1$ column vector where each entry is a 0 is denoted by $\mathbf{0}_m$.

It is relatively simple to calculate the well-covered dimension of a graph G, provided G is not too large. One first needs to find all the independent sets of G, which can be done using a greedy algorithm. Suppose that the maximal independent sets of G are M_i for $i = 0, \dots, k-1$. Then a well-covered weighting w of G is determined by a solution of the linear system of equations formed by selecting a maximal independent set, in this particular instance M_0, and setting the system $M_i - M_0 = 0$ for $i = 1, \dots, k-1$. This system is homogeneous, and can therefore be written in the form $A\mathbf{x} = \mathbf{0}$. Note that A is an $m \times n$ matrix where $m = k-1$ and $n = |V(G)|$. As this system is homogeneous, the nullity of A (here is where $char(\mathbf{F})$ could be relevant) is equivalent to $wcdim(G, \mathbf{F})$. So, $wcdim(G, \mathbf{F}) = n - rank(A)$. In the case when $n = rank(A)$, then $wcdim(G, \mathbf{F}) = 0$, which implies that in this case the only possible well-covered weighting is the 0 function.

For the remainder of this paper, we shall concern ourselves only with the determining of the well-covered dimensions for various individual graphs and graph families. We start by recalling a lemma from [1], as it will allow us to focus only on connected graphs.

Lemma 1 (Brown & Nowakowski [1])**.** *Let G and H be graphs. Then*

$$wcdim(G \cup H, \boldsymbol{F}) = wcdim(G, \boldsymbol{F}) + wcdim(H, \boldsymbol{F})$$

The family of (connected) graphs that is possibly the easiest to attack when looking for well-covered dimensions is the family of complete graphs. By simply looking at the maximal independent sets of K_n we get that $wcdim(K_n) = 1$. Similarly, using the previous lemma, we get $wcdim(\overline{K_n}) = n$.

2. The Well-covered dimension of certain families of graphs

Only using the technique mentioned above, it is easy to find the well-covered dimension of several families of graphs.

Example 1. *(a) $wcdim(Pe) = 0$, where Pe is the Petersen graph.*
(b) The two graphs in figure 1 below are he smallest graphs such that their well-covered dimension depends on the characteristic of the base field (SG_1) or it is zero (SG_2). In fact,

$$wcdim(SG_1, \boldsymbol{F}) = \begin{cases} 3 & \text{if } char(\boldsymbol{F}) = 2 \\ 2 & \text{otherwise} \end{cases}$$

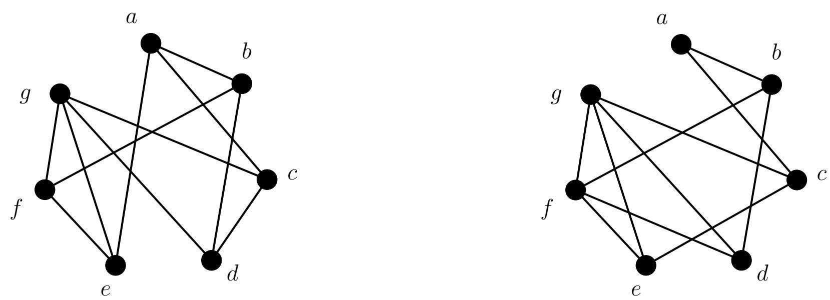

FIGURE 1. SG_1 (left) and SG_2 (right).

A similar result is obtained for crown graphs. Recall that, for any $n > 2$, the crown graph S_n^0 is formed by removing a perfect matching from $K_{n,n}$. Though not specifically stated as such, it was proven in [1] that $wcdim(S_n^0, \mathbf{F}) = n - 1$, if $char(\mathbf{F}) = 0$, and $wcdim(S_n^0, \mathbf{F}) = n$ if both $char(\mathbf{F})$ and n are even. We shall extend this result to allow us to calculate the well-covered dimensions of all crown graphs over all fields.

Theorem 1. *Let S_n^0 denote a crown graph, for all $n \in \mathbb{N}$. Then,*

$$wcdim(S_n^0, \boldsymbol{F}) = \begin{cases} n & \text{if } char(\boldsymbol{F}) = p \neq 0 \text{ and } p|(n-2) \\ n-1 & \text{otherwise} \end{cases}$$

Proof. Let K_{V_1,V_2} be the complete bipartite graph with $V_1 = \{a_1, \cdots, a_n\}$ and $V_2 = \{b_1, \cdots, b_n\}$ and let $a_1b_1, \cdots, a_nb_n$ be the perfect matching that is removed from K_{V_1,V_2} to form S_n^0. The maximal independent sets of S_n^0 are $\{a_i, b_i\}$ for $i = 1, 2, 3, \cdots, n$, and V_1 and V_2. Setting the sum of each of the weights on the maximal independent sets equal to that of the weights on the vertices of V_2, we find that the linear system corresponding to the well-covered weightings is $A\mathbf{x} = 0$, where

$$A = \begin{pmatrix} I_n & I_n - J_n \\ \mathbf{1}_n^T & -\mathbf{1}_n^T \end{pmatrix},$$

an $(n+1) \times 2n$ matrix. Subtracting the top n rows from the bottom yields

$$\begin{pmatrix} I_n & I_n - J_n \\ \mathbf{0}_n^T & (n-2)\,\mathbf{1}_n^T \end{pmatrix}.$$

It follows that we have two possibilities depending on whether or not $char(F)$ divides $n - 2$. The theorem follows after finding the rank of this matrix in either case. $\square$

Theorem 2. $wcdim(K_{|n_0|,\ldots,|n_{k-1}|}) = \sum_{i=0}^{k-1} |n_i| - (k-1)$, *where $K_{|n_0|,\ldots,|n_{k-1}|}$ is the (obvious) complete k-partite graph.*

Proof. Let f be a well-covered weighting of $K_{|n_0|,\ldots,|n_{k-1}|}$. The maximal independent sets of $K_{|n|,\ldots,|n_{k-1}|}$ are n_i for $i = 0, \ldots, k-1$. Setting the sum of each of the weights on the maximal independent sets equal to that of the weights on the vertices of n_{k-1}, we find that the linear system corresponding to the well-covered

weightings is $A\mathbf{x} = 0$, where

$$A = \begin{pmatrix} \mathbf{1}^T_{|n_0|} & \mathbf{0}^T_{|n_1|} & \mathbf{0}^T_{|n_2|} & \cdots & \mathbf{0}^T_{|n_{k-2}|} & -\mathbf{1}^T_{|n_{k-1}|} \\ \mathbf{0}^T_{|n_0|} & \mathbf{1}^T_{|n_1|} & \mathbf{0}^T_{|n_2|} & \cdots & \mathbf{0}^T_{|n_{k-2}|} & -\mathbf{1}^T_{|n_{k-1}|} \\ \mathbf{0}^T_{|n_0|} & \mathbf{0}^T_{|n_1|} & \mathbf{1}^T_{|n_2|} & \cdots & \mathbf{0}^T_{|n_{k-2}|} & -\mathbf{1}^T_{|n_{k-1}|} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}^T_{|n_0|} & \mathbf{0}^T_{|n_1|} & \mathbf{0}^T_{|n_2|} & \cdots & \mathbf{0}^T_{|n_{k-2}|} & -\mathbf{1}^T_{|n_{k-1}|} \end{pmatrix},$$

a $(k-1) \times (\sum_{i=0}^{k-1} |n_i|)$ matrix. A has rank $k-1$. Hence the nullity is $\sum_{i=0}^{k-1} |n_i| - (k-1)$, which implies that $wcdim(K_{|n_0|,\ldots,|n_{k-1}|}) = \sum_{i=0}^{k-1} |n_i| - (k-1)$. □

Corollary 1. *$wcdim\,(T\,(n,r)) = (n \bmod r)\,\lceil n/r \rceil + (r - (n \bmod r))\,\lfloor n/r \rfloor - (r-1)$, where $T(n,r)$ denotes a Turán graph.*

We can see immediately that if n is divisible by r, then corollary 1 reduces to $wcdim\,(T\,(n,r)) = n - (r-1)$.

The behaviors, in terms of well-covered weightings, of paths and cycles are very similar. Hence, we will study these two families simultaneously.

Consider G to be an n-path or an $(n+2)$-cycle, for $n \geq 6$. Label six 'consecutive' vertices a, b, c, d, e and f as in the picture below. Let w be a well-covered weighting of G, and let M_1 and M_2 be two maximal independent sets of G that contain the same vertices except from M_1 containing $\{a, c, f\}$ and M_2 containing $\{a, d, f\}$ instead. Locally, these two independent sets are represented in the figure below.

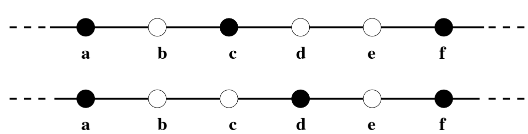

Figure 2. M_1 and M_2 on six consecutive vertices.

Since M_1 and M_2 just 'interchange' c and d, then $w(c) = w(d)$. It is now immediate that all vertices of C_n, for $n \geq 8$, have the same weight for all well-covered weightings of this graph. Hence, $wcdim(C_n) \leq 1$ for all $n \geq 8$.

Now consider two maximal independent sets N_1 and N_2 of C_n, with $n \geq 8$, that contain the same vertices outside of a string of seven consecutive vertices, where N_1 and N_2 contain four and three vertices respectively. These seven vertices, with the vertices contained in N_1 and N_2 are represented in figure 3 below.

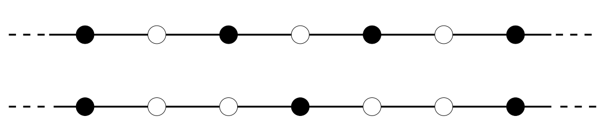

Figure 3. Two maximal independent sets with different cardinality.

It follows that this graph admits maximal independent sets with different cardinalities, and thus $wcdim(C_n) = 0$ for all $n \geq 8$.

Similarly, from the argument associated to figure 2, if $n \geq 6$ and $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ (edges connecting v_i with v_{i+1}) then vertices $v_3, \cdots, v_{n-2}$ must have the same weight for all well-covered weightings of P_n. Moreover, for small values of n it is easy to see that these weights must be zero. For larger values of n figure 3 provides a way to construct maximal independent sets with different cardinality, which forces $w(v_3) = \cdots = w(v_{n-2})$.

Finally, we can construct two maximal independent sets of P_n that share all but one vertex, which is v_1 for one of them and v_2 for the other. This can be seen in the figure below.

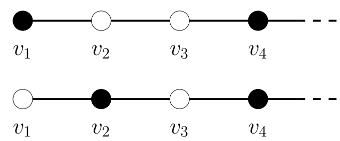

FIGURE 4. Two maximal independent sets on the first four vertices.

It follows that $w(v_1) = w(v_2)$, and symmetrically that $w(v_{n-1}) = w(v_n)$, for all well-covered weightings w of P_n. Lastly, we want to remark that $w(v_1)$ is independent of $w(v_n)$, and thus, adding simple computations to the arguments above we obtain the following theorem:

Theorem 3. *If w is a well-covered weighting of P_n and $n \geq 5$, then $w(v_1) = w(v_2)$, $w(v_3) = \ldots = w(v_{n-2}) = 0$, and $w(v_{n-1}) = w(v_n)$. Moreover, $wcdim(P_2) = 1$ and $wcdim(P_n) = 2$ if $n > 1$.*

Also,

(i) $wcdim(C_n) = 0$, *if* $n \geq 8$,
(ii) $wcdim(C_n) = 1$, *if* $n = 3, 5, 7$,
(iii) $wcdim(C_6) = 2$,
(iv) $wcdim(C_4) = 3$.

Remark 1. *The well-covered dimensions of paths had been already computed in* [3] *using methods different from the ones used in this paper.*

Now we look at the family of gear graphs. A gear graph over $2n+1$ vertices, denoted G_n is the graph with vertex set $V(G_n) = \{v_0, ..., v_{2n-1}, v_c\}$ where:

(1) v_i is adjacent to $v_{i-1 \bmod 2n}$ and $v_{i+1 \bmod 2n}$ for $0 \leq i \leq 2n-1$.
(2) if $i \in 2\mathbb{Z}$, then v_i is adjacent to v_c.

We can compute the well-covered dimensions of the gear graphs using the same methods we used to compute the well-covered dimensions of the cycle graphs.

Corollary 2. *Let G_n be the gear graph in $2n+1$ vertices, then*

$$wcdim(G_n) = \begin{cases} 3 & \text{if } n = 3 \\ 0 & \text{if } n > 3 \end{cases}$$

3. BLOWUPS AND LEXICOGRAPHIC PRODUCTS

In this section we look at the well-covered dimension of graphs that can be constructed from known ones by using various techniques. We begin with a definition.

Definition 1. *Let G be a graph and $t \in \mathbb{N}$. A t-blowup of a vertex $v_i \in V(G)$ is an independent set $V_{v_i} = \{v_{i1}, v_{i2}, \cdots, v_{it}\}$ that 'takes the place' of v. More precisely, wherever there was an edge joining v to $w \in V(G)$ there is an edge joining v_i with w.*

The graph obtained by the t-blowup of v will be denoted $G(tv)$. Similarly, for $v, w \in V(G)$ and $s, t \in \mathbb{N}$ we denote a 'double blowup' $G(tv)(sw)$ as $G(tv, sw)$. For multiple blowups we extend in the natural way the notation set of double blowups.

Note that $G(v) = G$ for all $v \in V(G)$.

Lemma 2. *Let G be a graph with $V(G) = \{v_1, \cdots, v_n\}$, and $m = wcdim(G, \boldsymbol{F})$. Let $H = G(tv_1)$, where $t \in \mathbb{N}$. Then, $wcdim(H, \boldsymbol{F}) = m + t - 1$.*

Proof. We begin by noticing that a maximal independent set of G not containing v_1 is also a maximal independent set of H, and if $S = \{v_1, v_{i_2}, \cdots, v_{i_r}\}$ is a maximal independent set of G then

$$S' = \{v_{11}, \cdots, v_{1t}, v_{i_2}, \cdots, v_{i_r}\}$$

is a maximal independent set of H. Moreover, it is easy to see that every maximal independent set of H must be of one of these two types.

Let M and $M(t)$ be the matrices associated to the systems of equations arising from looking for well-covered weightings of G and H respectively. We notice that $M(t)$ has $t-1$ more columns than M but that it has exactly the same number of rows, and in fact the same rank as M, which is $n - m$. The result follows. □

By using this lemma repeatedly in a graph that is constructed from G by a sequence of blowups of vertices of G we get the following theorem.

Theorem 4. *Let G be a graph with $V(G) = \{v_1, \cdots, v_n\}$ and $m = wcdim(G, \boldsymbol{F})$. Let $H = G(t_1v_1, t_2v_2, \cdots, t_nv_n)$, where $t_i \in \mathbb{N}$ for all $i = 1, 2, \cdots, n$. Then,*

$$wcdim(H, \boldsymbol{F}) = (m - n) + \sum_{i=1}^{n} t_i$$

Now we will look at the lexicographic product of graphs. We start with a definition.

Definition 2. *The lexicographic product of G and H, denoted $G \bullet H$, is the graph with vertex set $V(G) \times V(H)$ and edges joining (g, h) and (g', h') if and only if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$.*

Corollary 3. *Let G be a graph in n vertices with $wcdim(G, \boldsymbol{F}) = m$. Then,*

$$wcdim\left(G \bullet \overline{K_t}, \boldsymbol{F}\right) = m + n(t-1)$$

where $t \in \mathbb{N}$.

Proof. Assume that $V(G) = \{v_1, \cdots, v_n\}$. The result follows from the previous theorem and the fact that $G(tv_1, tv_2, \cdots, tv_n) \cong G \bullet \overline{K_t}$. □

The previous corollary is also a corollary of theorem 5. In order to prove this theorem we need a couple of linear algebra results that we will not prove but will mention in full detail.

Lemma 3. *Let M be an $n \times m$ matrix and let N be the $(n-1) \times m$ matrix obtained by subtracting the first row R_1 of M from all the other rows of M, and then deleting R_1. Assume $rank(N) = k$, then,*

$$rank(M) = \begin{cases} k & \text{if } R_1 \text{ is dependent of other rows of } M \\ k+1 & \text{if } R_1 \text{ is independent from other rows of } M \end{cases}$$

For the next couple of results, we denote the Kronecker (or tensor) product of two matrices, M and A, by $M \otimes A$.

Remark 2. *Let N, B and C be matrices obtained by using the construction described in lemma 3 from matrices M, A, and $M \otimes A$ respectively, we will re-arrange rows in the matrices if necessary to get, when possible, the first row to be dependent of the others. Then, $rank(C) = rank(M \otimes A)$ whenever there is a row that is dependent of others in A or M, as in these cases we can always choose a row of $M \otimes A$ that depends on the other rows of this matrix. On the other hand, if both M and A have linearly independent rows, then $M \otimes A$ also has this property (using $rank(A \otimes M) = rank(A)rank(M)$), and thus $rank(C) = rank(M \otimes A) - 1$.*

If we now use lemma 3, and assume $rank(N) = k$ and $rank(B) = q$, then

$$rank(C) = \begin{cases} kq & \text{if both } M \text{ and } A \text{ have linearly dependent rows} \\ k(q+1) & \text{if } M \text{ has linearly dependent rows and } A \text{ does not} \\ (k+1)q & \text{if } A \text{ has linearly dependent rows and } M \text{ does not} \\ (k+1)(q+1)-1 & \text{if both } M \text{ and } A \text{ have linearly independent rows} \end{cases}$$

Now we have all the tools needed to prove.

Theorem 5. *Let G and H be graphs with $|V(G)| = a$, $|V(H)| = b$, $wcdim(G, \boldsymbol{F}) = n$, and $wcdim(H, \boldsymbol{F}) = m$. Then,*

$$wcdim(G \bullet H, \boldsymbol{F}) = nb + am - nm + \delta_{m-b+1,i}(n-a) + \delta_{n-a+1,j}(m-b)$$

where δ_{xy} represents the Kronecker delta, and i, j are the number of maximal independent sets of H and G respectively.

Proof. We first notice that if $S = \{v_{1_1}, v_{i_2}, \cdots, v_{i_r}\}$ is a maximal independent set of G then

$$S' = \{w_{1i_1}, \cdots, w_{t_1 i_1}, w_{1i_2}, \cdots, w_{t_2 i_2}, \cdots, w_{1i_r}, \cdots, w_{t_r i_r}\}$$

is a maximal independent set of $G \bullet H$, where $\{w_{1i_j}, \cdots, w_{t_j i_j}\}$ is a maximal independent set of H for all $j = 1, 2, \cdots, r$. Moreover, it is easy to see that every maximal independent set of $G \bullet H$ must be obtained this way.

Set the weight-sums of each of the independent sets of G equal to zero. Let M be the matrix representing that homogeneous system of equations. Note that the matrix N needed to find $wcdim(G, \mathbf{F})$ is obtained from M by using the construction described in lemma 3. Similarly, by repeating this process with H we obtain B (needed for finding $wcdim(H, \mathbf{F})$) out of A (found by setting the weight-sums of the maximal independent sets of H equal to zero).

Now we notice that (because of the first paragraph in this proof) $A \otimes M$ is the matrix associated to the homogeneous system given by setting the weight-sums of all the independent sets of $G \bullet H$ equal to zero. It follows that we are interested in finding the rank of the matrix C obtained from $A \otimes M$ by using the construction described in lemma 3.

Since $rank(N) = a - n$, $rank(B) = b - m$, and $|V(G \bullet H)| = ab$, then using that a matrix has linearly dependent rows if and only if its rank is not equal to its number of rows, and remark 2, we get

$$wcdim(G \bullet H, \mathbf{F}) = \begin{cases} nb + am - nm & \text{if } i \neq b - m + 1,\ j \neq a - n + 1 \\ nb + am - nm + n - a & \text{if } i = b - m + 1,\ j \neq a - n + 1 \\ nb + am - nm + m - b & \text{if } i \neq b - m + 1,\ j = a - n + 1 \\ nb + am - nm + m - b + n - a & \text{if } i = b - m + 1,\ j = a - n + 1 \end{cases}$$

where i, j represent the number of maximal independent sets of H and G respectively (which are the number of rows of A and M respectively).

The result follows from the definition of the Kronecker delta. □

Corollary 4. *Let G and H be graphs with more maximal independent sets than vertices, and such that $|V(G)| = a$, $|V(H)| = b$, $wcdim(G, \boldsymbol{F}) = n$ and $wcdim(H, \boldsymbol{F}) = m$. Then,*

$$wcdim(G \bullet H, \boldsymbol{F}) = nb + am - nm$$

Remark 3. *As mentioned above, corollary 3 is a corollary of theorem 5. In order to see this we just need to notice that $\overline{K_t}$ has one maximal independent set and that $wcdim(\overline{K_t}) = V(\overline{K_t}) = t$.*

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